

MAS212 Linear Algebra I 2007-2008

Coursework 8

Please hand in your solutions of the **starred** exercises by **2pm on Friday 30 November** using the [blue Linear Algebra I Collection Box](#) on the Ground Floor.

Don't forget to put your **name** and **student number** on your solutions, and to **staple** them.

Exercise* 1. (6 marks) Let C be a fixed $n \times n$ matrix. Determine which of the following are linear transformations from $\mathbb{R}^{n \times n}$ to $\mathbb{R}^{n \times n}$:

- (a) $L(A) = 3A$, (b) $L(A) = A^T + A$, (c) $L(A) = A - I$,
(d) $L(A) = CA - AC$, (e) $L(A) = C^2A$, (f) $L(A) = A^2C$.

Exercise* 2. (2 marks) Determine which of the following are linear transformations from P_2 to P_2 :

- (a) $(L(p))(t) = t + p(t)$, (b) $(L(p))(t) = tp'(t) + p''(t)$.

Exercise* 3. (2 marks) A linear transformation $L : V \rightarrow W$ is said to be **one-to-one** if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies that $\mathbf{v}_1 = \mathbf{v}_2$ (that is, no distinct vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ get mapped to the same vector $\mathbf{w} \in W$). Show that L is one-to-one if and only if $\ker(L) = \{\mathbf{0}\}$.

Exercise* 4. (3 marks) For each of the following linear transformations find its matrix representation with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$:

- (a) $L : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ where $L(\mathbf{x}) = (x_1 + x_2, x_2, x_1 - x_2)^T$;
(b) $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ where $L(\mathbf{x}) = (-x_3, x_2 + 2x_3, x_3 - 4x_2)^T$;
(c) $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ where $L(\mathbf{x}) = (x_2 + x_3, x_1 - x_2 + 7x_3)^T$.

Exercise* 5. (2 marks) Let $\mathcal{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$ be the ordered basis of $\mathbb{R}^3$ given in Exercise 1 on Coursework 7, that is,

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} -3 \\ 1 \\ 12 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} 2 \\ 0 \\ -7 \end{pmatrix},$$

and let $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by $L(\mathbf{x}) = (x_2, x_3 - x_1, 2x_1 + x_3)^T$.

- (a) Find the matrix representation of L with respect to the standard basis of $\mathbb{R}^3$.
(b) Using (a) and the results from Exercise 3 (b) on Coursework 7, determine the matrix representation of L with respect to $\mathcal{B}$.

Exercise* 6. (5 marks) Let $L : P_2 \rightarrow P_2$ be the linear transformation given by

$$(L(p))(t) = tp'(t) + p''(t).$$

Let $\mathcal{P} = [\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3]$ and $\mathcal{Q} = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3]$ be the ordered bases of P_2 given by

$$\mathbf{p}_1(t) = 1, \quad \mathbf{p}_2(t) = t, \quad \mathbf{p}_3(t) = t^2, \quad \text{and} \quad \mathbf{q}_1(t) = 1, \quad \mathbf{q}_2(t) = t, \quad \mathbf{q}_3(t) = 1 + t^2.$$

- (a) Find the matrix representation of L with respect to $\mathcal{P}$.
(b) Find the transition matrix S from $\mathcal{Q}$ to $\mathcal{P}$ and the transition matrix from $\mathcal{P}$ to $\mathcal{Q}$.
(c) Using (a) and (b), determine the matrix representation of L with respect to $\mathcal{Q}$.
(d) If $\mathbf{p} \in P_2$ is given by $\mathbf{p}(t) = \alpha_1 + \alpha_2 t + \alpha_3(1 + t^2)$, find $L^n(\mathbf{p})$.

[Prize question overleaf.]

A cash prize is offered for the best solution received to the following question. Do not hand it in with your coursework, but hand it to me in person either at the Friday lecture or before.

Prize question. Let $L : V \rightarrow V$ be a linear transformation, and let S be a subspace of V . The subspace S is said to be an **invariant subspace** (of L) if $L(S) \subset S$. Clearly, the trivial subspaces of V , that is, V itself and $\{0\}$ are examples of invariant subspaces, but L may have other non-trivial invariant subspaces.

Let $A \in \mathbb{R}^{2 \times 2}$ be given by

$$A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}.$$

Find all invariant subspaces of $L_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

OB 22/11/2007